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Spanning trees with constraints on the leaf degree

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Abstract

Let T be a tree and m be a positive integer. The leaf degree of a vertex $x \in V(G)$ is defined as the number of end-vertices in T adjacent to x, and it is denoted by $\mathrm{leaf}_T x$. (If x is an end-vertex of T with at least three vertices, then $\mathrm{leaf}_T x = 0$.) We prove that a connected graph G has a spanning tree T such that for any vertex x of T, $\mathrm{leaf}_T x \leqslant m$ if and only if for every nonempty subset S of $V(G)$, the number of isolated vertices of $G - S$ does not exceed $(m+1)|S| - 1$.

Keywords: Graph; Factor; Spanning tree

1. Introduction

In this paper, we discuss finite undirected graphs with neither loops nor multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. Let S be a subset of $V(G)$. Denote by $G[S]$ the subgraph of G induced by the vertex set S, i.e., the graph having vertex set S and whose edge set consists of those edges of G incident with two elements of S. (If S is a subgraph of G, then we simply write $G[S]$ for $G[V(S)]$.) Let $K_{1,m}$ be the star with m end-vertices. A $\{K_{1,i} \mid 1 \leqslant i \leqslant m\}$*—factor* F of G is a spanning subgraph of G such that each component of F is isomorphic to $K_{1,i}$ for some i, $1 \leqslant i \leqslant m$. Let $i(G)$ be the number of isolated vertices of G.

In [1], Amahashi and Kano proved the following theorem.

Theorem A (Amahashi and Kano [1]). *Let G be a graph and m be an integer such that $m \geqslant 2$. Then G has a $\{K_{1,i} \mid 1 \leqslant i \leqslant m\}$—factor if and only if for every subset*

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S of $V(G)$,

$$i(G-S) \leqslant m|S|.$$

Let T be a tree. *The leaf degree* of a vertex $x \in V(T)$ is defined as the number of end-vertices in T adjacent to x, and it is denoted by $\mathrm{leaf}_T x$. (If x is an end-vertex of T with at least three vertices, then $\mathrm{leaf}_T x = 0$.) *The maximum leaf degree* of G is the maximum leaf degree among the vertices of G. A graph H is said to be *a triangle-tree* if it satisfies the following conditions:

(i) H is connected and every cycle of H, if any, is K_3, i.e., every block of H is an edge or a cycle with three vertices.
(ii) No two cycles of H have a vertex in common.

Note that a tree is also a triangle-tree.

The purpose of this paper is to prove the following necessary and sufficient condition for a connected graph to have a spanning tree T with maximum leaf degree at most m.

Theorem 1. *Let G be a connected graph and m be a positive integer. Then G has a spanning tree T with maximum leaf degree at most m if and only if for every nonempty subset S of $V(G)$,*

$$i(G-S) \leqslant (m+1)|S| - 1.$$

2. Proof of Theorem 1

Let T be a spanning tree T such that $\mathrm{leaf}_T x \leqslant m$ for any vertex x of T. Assume, to the contrary, that there exists a nonempty subset S of $V(G)$ such that $i(G-S) \geqslant (m+1)|S|$. Thus let A be a set of isolated vertices of $G-S$ such that $|A| \geqslant (m+1)|S|$. Note that every edge incident to a vertex in A must be incident to some vertex in S. Let W be the set of those vertices in A whose degree in T is at least two. Obviously, every edge in T incident to a vertex in W must be incident to some vertex in S. Thus, since the maximum leaf degree of T is at most m, we have $|W| \geqslant |S|$. Since $\mathrm{leaf}_T x \geqslant 2$ for any vertex x in W, we see that $T[W \cup S]$ has at least $2|W|$ edges. Since $|W| \geqslant |S|$, we know that $T[W \cup S]$ has at least $|W| + |S|$ edges, which implies that $T[W \cup S]$ has a cycle, a contradiction. This establishes the necessity.

Thus, $m|S| \geqslant |A - W| = |A| - |W| \geqslant (m+1)|S| - |W|$ by $|A| \geqslant (m+1)|S|$ and we have $|W| \geqslant |S|$. Since the degree of any vertex in W is at least two in the tree T, we see that $T[W \cup S]$ has at least $2|W|$ edges. Since $|W| \geqslant |S|$.

Next, we consider the sufficiency. Let W and X be subgraphs in G (possibly $V(W)=\emptyset$ or $V(X)=\emptyset$). Then a pair (W,X) is called *an m-admissible pair* of G if it satisfies the following conditions:

(i) $V(W) \cap V(X) = \emptyset$ and $V(W) \cup V(X) = V(G)$.
(ii) Each component of X is isomorphic to $K_{1,m+1}$.

(iii) $|V(R)| \geqslant 2$ for each component R of W.
(iv) If $m \geqslant 2$, then each component of W is a tree with maximum leaf degree at most m.
(v) If $m = 1$, then each component of W is a triangle-tree H with maximum leaf degree at most one such that $\mathrm{leaf}_H z = 0$ for any vertex z contained in a cycle C (i.e., K_3) in H.

First, we show the existence of an m-admissible pair of G. Since for every nonempty subset S of $V(G)$, $i(G-S) \leqslant (m+1)|S| - 1 < (m+1)|S|$, it follows from Theorem A that G has a $\{K_{1,i} \mid 1 \leqslant i \leqslant m+1\}$—factor F. Let W be the set of all components $K_{1,j}$ of F such that $1 \leqslant j \leqslant m$. Set $X = F - V(W)$. Clearly, a pair (W,X) is an m-admissible pair of G.

Among all m-admissible pairs of G, let (W,X) be one such that $|V(W)|$ is as large as possible. We claim that W is a spanning subgraph of G, i.e., $V(X) = \emptyset$. Suppose, to the contrary, that $V(X) \neq \emptyset$. Let $C_1, C_2, \ldots, C_k$ be the components of X, where C_i is isomorphic to the star $K_{1,m+1}$ for all i, $1 \leqslant i \leqslant k$. Let s_i be the center of the star C_i and let $S = \{s_1, s_2, \ldots, s_k\}$. Hence $X - S$ consists of $(m+1)k$ isolated vertices. By the assumption, we have $i(G-S) \leqslant (m+1)k - 1$. Therefore, it follows that in G there is an edge joining a vertex x of $X - S$, say $x \in V(C_1) - s_1$, to a vertex y of $W \cup (X - (S \cup \{x\}))$.

If the vertex y belongs to some component R of W, then $R \cup xy \cup C_1$ is a tree or a triangle-tree with maximum leaf degree at most m. Set $H = R \cup xy \cup C_1$. Since the pair (W,X) is an m-admissible pair of G, if H is a triangle-tree, then by the assumption, $\mathrm{leaf}_R z = 0$ for any vertex z contained in a cycle C in R, so that $\mathrm{leaf}_H z = 0$ for any vertex z contained in a cycle C in H. Set $W' = W \cup xy \cup C_1$ and $X' = X - V(C_1)$. Obviously, the pair (W',X') is an m-admissible pair of G such that $|V(W)| < |V(W')|$, which contradicts the maximal property of W.

Thus the vertex y belongs to $X - (S \cup \{x\})$. If $y \in V(C_i)$ for some i, $2 \leqslant i \leqslant k$, then $C_1 \cup xy \cup C_i$ is a tree with maximum leaf degree at most m. Set $W' = W \cup C_1 \cup xy \cup C_i$ and $X' = X - (V(C_1) \cup V(C_i))$. Again the pair (W',X') is an m-admissible pair of G such that $|V(W)| < |V(W')|$, which contradicts the maximal property of W.

Hence, we see that $y \in V(C_1) - \{x, s_1\}$. If $m \geqslant 2$, then $C_1 \cup xy - xs_1$ is a tree with maximum leaf degree at most m and set $H = C_1 \cup xy - xs_1$. If $m = 1$, then $C_1 \cup xy$ is a cycle with three vertices and set $H = C_1 \cup xy$. In either case, set $W' = W \cup H$ and $X' = X - V(C_1)$. It is easy to check that the pair (W',X') is an m-admissible pair of G such that $|V(W)| < |V(W')|$, which contradicts the maximal property of W. Thus $V(X) = \emptyset$ and W is a spanning subgraph of G, as claimed.

If $m \geqslant 2$, then since G is connected, it is easy to see that W, together with suitable edges of G, form a spanning tree of G whose maximum leaf degree is at most m. If $m = 1$, then by the same argument we obtain a spanning triangle-tree H with maximum leaf degree at most one such that $\mathrm{leaf}_H z = 0$ for any vertex z contained in a cycle C (i.e., K_3) in H. By deleting a suitable edge from each cycle of H, we can obtain a spanning tree of G whose maximum leaf degree is (at most) one.

This completes the proof of Theorem 1. □

3. Concluding remarks

Let T be a tree and let u and v be two vertices in T. Denote by $END(T)$ the set of all end-vertices of T. Denote by $d_{\mathrm{T}}(u,v)$ the length of the path joining u to v in T. Set

$$lr(T)=\min\{d_{\mathrm{T}}(u,v)\colon u\in END(T),\ v\in END(T),\ u\neq v\}.$$

$lr(T)$ is called *the leaf radius* of T. For example, if T is a tree with at least three vertices, then $lr(T)=2$ when the maximum leaf degree of T is at least two, and $lr(T)\geqslant 3$ when the maximum leaf degree of T is at most one. On the other hand, let $K(1),K(2),\dots,K(k)$ be k disjoint copies of the complete graph $K_{d/2}$ and $K(k+1)$ be a copy of the complete graph $K_{d/2-1}$. Take a new vertex x and join x to all vertices of

$$K(1)\cup K(2)\cup\cdots\cup K(k+1).$$

Denote the resulting graph by $H(d)$. It is easy to check that $lr(T)\leqslant d-1$ for every spanning tree T of $H(d)$ and that there is a spanning tree of $H(d)$ where equality holds. Let v_i be a vertex of $K(i)$, $1\leqslant i\leqslant k+1$. Set $S=V(H(d))-\{v_1,v_2,\dots,v_{k+1}\}$, so that $|S|=(k+1)(\frac{d}{2}-1)$. Since $\{v_1,v_2,\dots,v_{k+1}\}$ is an independent set of $H(d)$, $i(H(d)-S)=|\{v_1,v_2,\dots,v_{k+1}\}|=k+1=|S|/(d/2)-1=2|S|/d-2$. Thus the condition in the following conjecture is in some sense the best possible.

Conjecture B. *Let G be a connected graph and d be an integer such that $d\geqslant 4$. If for every nonempty subset S of $V(G)$*

$$i(G-S)<\frac{2|S|}{d-2},$$

then G has a spanning tree T such that the leaf radius of T is at least d.

Notice that by Theorem 1 the statement is true for the case $d=3$.

References

[1] A. Amahashi, M. Kano, Factors with given components, Discrete Math. 42 (1982) 1–6.